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Gel'fand bases and the permutation representations of the symmetric group associated with the subgroups $S_{\lambda 1}\times S_{\lambda 2}\times\ldots\times S_{\lambda n}$

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Abstract. A complete decomposition of the space of tensors of rank k over a vector space of dimension n into basis vectors transforming canonically under $S_k\times GL(n)$ is performed using only the properties of the symmetric group. The basis arises in a natural way (via Frobenius reciprocity) by identifying each subspace of tensors of a fixed weight λ with the permutation representation of S_k induced by the subgroup $S_\lambda = S_{\lambda 1}\times S_{\lambda 2}\times\ldots\times S_{\lambda n}$.

1. Introduction

In spite of the well-known relationship between the irreducible representations of the symmetric groups S_k and the general linear groups $GL(n)$, it has been only comparatively rarely that the connection has been used by authors at the level of detail where one works with explicit bases. The 'dual' nature of the tensor representations of $GL(n)$ and S_k enable the representing matrices to be decomposed into block form according to the irreducible components of S_k (see, for instance, Weyl 1946). However, when it is desired to go further and actually find a complete set of basis vectors corresponding to each block, generally speaking the 'duality' is allowed to fall into the background: one either uses Young tableaux to form projectors (as described, for instance, in the books of Weyl (1946), Boerner (1968) or Hammermesh (1962)), or else, as in the method of fractional parentage (e.g. Judd 1963) a decomposition is effected under subgroups of $GL(n)$ which bear little relation to the symmetric group.

A recent paper by Patterson and Harter (1976), developing some earlier results of Lezuo (1972), has shown that this need not always be the case. The Gel'fand basis of $GL(n)$, defined with the help of the chain

$$GL(n)\supset GL(n-1)\supset\ldots\supset GL(1),$$

may be pinpointed, it turns out, in tensor representations by projectors derived purely from the symmetric group.

Associated with this is a well-known method for enumerating Gel'fand basis vectors using Young tableaux. By a 'partition λ of k into n parts', we shall mean an ordered n-tuple of non-negative integers $(\lambda_1,\ldots,\lambda_n)$ satisfying $\Sigma\lambda_i = k$. Let $\lambda=(\lambda_1,\ldots,\lambda_n)$ and $\mu=(\mu_1,\ldots,\mu_n)$ be two partitions of k into n parts and also let μ be 'dominant', that is, $\mu_1\geqslant\mu_2\geqslant\ldots\geqslant\mu_n$. By a $[GL(n)]$ *Gel'fand pattern* we shall mean a triangular array of integers $\{m_{ij};\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant i\}$ satisfying the 'betweenness conditions'

$$m_{ij} \geqslant m_{i-1j} \geqslant m_{ij+1}.$$

Such a pattern labels a basis vector of weight λ in the $\mathrm{GL}(n)$ irreducible representation with highest weight μ provided that

$$m_{nj} = \mu_j \qquad j = 1, \ldots, n$$

$$\sum_{j=1}^{i+1} m_{i+1j} - \sum_{j=1}^{i} m_{ij} = \lambda_{i+1} \qquad i = 1, \ldots, n-1 \qquad m_{11} = \lambda_1.$$

It turns out (Boerner 1968, Baird and Biedenharn 1963, Ciftan and Biedenharn 1969) that the set of standard Young tableaux of shape μ that can be formed using λ_1 1's, λ_2 2's, $\ldots$, λ_n n's is in one-to-one correspondence with the set of Gel'fand patterns satisfying both the above conditions.

Another set whose elements may be enumerated by the same set of standard tableaux lies at the heart of the technique we shall present in this paper for decomposing spaces of tensors. If ρ^ϵ is the (permutation) representation of S_k induced by the identity representation ϵ of the subgroup

$$\mathrm{S}_\lambda = \mathrm{S}_{\lambda 1} \times \ldots \times \mathrm{S}_{\lambda n},$$

the multiplicity with which the IR μ of S_k occurs in ρ^ϵ is given by the number of elements in the above set (see, for example, Littlewood 1940, Robinson 1961, Coleman 1966). We shall identify each space of tensors as a direct sum of such representations and show, using results derived in § 2, how a generalisation of the Frobenius reciprocity theorem gives rise to an explicit basis for each summand. The explicit basis arises (via reciprocity) from the notion, introduced by Elliott *et al* (1953) and studied extensively by Kaplan (1962, 1975), of non-standard bases for subduced representations of S_λ by S_k. The non-standard basis vectors corresponding to the identity representation of S_λ in the IR μ of S_k are seen to be labelled by Gel'fand patterns. This gives rise to a reduction of carrier spaces of the representations ρ^ϵ and so also of the appropriate tensor spaces. The basis vectors resulting from this construction are already labelled by Gel'fand patterns and are shown in fact to have the correct transformation properties under subgroups of $\mathrm{GL}(n)$. The expression finally arrived at for these vectors is quite similar to that found by Patterson and Harter *ab initio*, and the exact relationship between that paper and the present one is discussed.

2. Reciprocal basis sets

Let G be a finite group and H one of its subgroups, and let $\mu : \mathrm{G} \to \mathrm{End}\ V$, $\lambda : \mathrm{H} \to \mathrm{End}\ W$ be irreducible representations of G and H, respectively. We can assume that V and W are equipped with inner products so that the matrices of each representation are unitary. Let $D_\mu = \dim V$, $D_\lambda = \dim W$.

A subduced representation of H, which we shall write as $\mu|_{\mathrm{H}}$, is defined by restricting μ from G to H. This representation is no longer in general irreducible, and to analyse it it is convenient to introduce the space $M = \mathrm{Hom}_{\mathrm{H}}(W, V)$ of operators from W into V which intertwine with the two representations λ and $\mu|_{\mathrm{H}}$ of H. If $A \in M$, $A \neq 0$, then AW is a subspace of V carrying an irreducible representation of H equivalent to λ. M can be equipped with an inner product with the properties that $\langle A, A\rangle = 1 \Rightarrow A$ unitary and $A \perp B \Rightarrow AW \perp BW$. It is not hard to see this is realised by defining

$$\langle A, B\rangle = D_\lambda^{-1}\,\mathrm{Tr}(A^*B).$$

The problem of finding an orthogonal labelling scheme for resolving the multiplicity of λ in μ is then equivalent to that of finding an orthogonal basis for M. To be more explicit, let $\{w_s;\ s = 1, D_\lambda\}$ be an ONB for W and let $\{A_\alpha;\ \alpha = 1, \text{multiplicity of } \lambda \text{ in } \mu\}$ be an ONB for M. Then $\{A_\alpha w_s\}$ is an orthonormal basis of MW, which is the component of V transforming under $\mu|_{\mathrm{H}}$ as λ.

Consider now the representation ρ^λ of G induced by the IR λ of H. (For the fundamentals of induced representations see, for instance, Mackey 1968.) It is defined on the space L of functions $\psi : \mathrm{G} \to W$ which satisfy

$$\psi(gh) = \lambda(h^{-1})\psi(g) \qquad g \in \mathrm{G},\ h \in \mathrm{H}.$$

The operator representing $g \in G$ is given by

$$\rho^\lambda(g)\psi(g') = \psi(g^{-1}g') \qquad g' \in \mathrm{G}$$

and the inner product on L is defined in terms of that on W by

$$\langle \psi, \phi\rangle = |G|^{-1} \sum_{g\in G} \langle \psi(g), \phi(g)\rangle.$$

The problem of resolving the multiplicity of μ in ρ^λ can be formulated in just the same way as the problem of λ in $\mu|_{\mathrm{H}}$. Let $N = \mathrm{Hom}_{\mathrm{G}}(V, L)$ be the space of G-intertwining operators $V \to L$. Thus if $A \in N$, AV is a subspace of L carrying a representation of G equivalent to μ. Defining an inner product analogously to that on M, we may then resolve the multiplicity by selecting an ONB $\{B_\alpha\}$ of N. If $\{v_r;\ r = 1, D_\mu\}$ is an arbitrary ONB of V, then $\{B_\alpha v_r\}$ is an ONB of NV, the subspace of L transforming under ρ^λ like μ.

One way of stating the Frobenius reciprocity theorem is to say that the spaces M and N have the same dimension. A proof of the theorem is then obtained by constructing an antilinear mapping

$$T : M \to N$$

$$A \mapsto T_A$$

given by

$$T_A v(g) = A^*\mu(g^{-1})v \qquad v \in V,\ g \in \mathrm{G}$$

and showing that it is bijective (Gaal 1973, Kirillov 1976, Ol'sanskii 1969). One can in fact straightforwardly show (appendix 1) that

$$\langle T_A, T_B\rangle = D_\lambda D_\mu^{-1}\langle B, A\rangle$$

so that the operator $\hat{T} = (D_\mu D_\lambda^{-1})^{1/2}T$ is antiunitary. Bearing in mind that orthonormal bases of M and N define orthonormal labelling schemes for copies of λ in V and copies of μ in L, one sees immediately the labelling problems for subduced and induced representations are equivalent: a solution to one problem defines a solution to the other.

The basis-independent definition of T given above disguises the fact that a familiar form can be given to the labelling scheme for L corresponding to a given reduction of μ. The components of any element of L along basis elements of W define functions $\mathrm{G} \to \mathbb{C}$

$$\psi_s(g) = \langle w_s, \psi(g)\rangle \qquad \text{for any } \psi \in L.$$

Imagine now that the bases $\{A_\alpha\}$ of M, $\{v_r\}$ of V, and $\{w_s\}$ of W are given. Then the component of $\chi_{\alpha r} = \hat{T}_{A_\alpha} v_r$ along w_s is simply

$$\langle w_s, \chi_{\alpha r}(g)\rangle = (D_\mu D_\lambda^{-1})^{1/2} \langle \mu(g) A_\alpha w_s, v_r\rangle.$$

The components of $\chi_{\alpha r}$ are thus matrix element functions for the representation μ between appropriately chosen basis elements. The two bases

$$\{u_{\alpha s} = A_\alpha w_s\} \qquad \text{of } MW$$

and

$$\{\chi_{\alpha r} = \hat{T}_{A_\alpha} v_r\} \qquad \text{of } NV$$

can be called 'reciprocal'. The antiunitarity of $\hat{T}$ implies that if one is orthonormal so is the other.

3. Decomposition of tensor spaces

3.1. Tensors as permutation representations of S_k

A *permutation* representation of a group G is one induced by the identity representation of a subgroup H. These arise whenever one considers a space $L^2(X)$ of complex-valued functions on X where X is a set on which G acts transitively. A representation π_X of G is defined on $L^2(X)$ by

$$\pi_X(g)\beta(x) = \beta(g^{-1}x) \qquad g \in \mathrm{G}, x \in X$$

where we use $gx \in X$ to denote the action of G on X. For a fixed $x \in X$, the stabiliser subgroup S_x of x is the subgroup of G that fixes x. Provided the action of G on X is transitive the representation π_X may be identified with the permutation representation ρ^ϵ of G induced by the identity representation of S_x. When G does not act transitively on X, we must first decompose X into disjoint orbits: let O_x be the set

$$\mathrm{O}_x = \{gx; g \in \mathrm{G}\}.$$

One can always find a subset T of X which 'traverses' X; in other words

$$X = \bigcup_{x \in T} \mathrm{O}_x$$

$$\mathrm{O}_x \cap \mathrm{O}_{x'} = \phi \qquad \text{if } x \neq x', x, x' \in T.$$

The representation π_X is then a direct sum of permutation representations corresponding to the decomposition

$$L^2(X) = \bigoplus_{x \in T} L^2(\mathrm{O}_x).$$

To apply this procedure to the space $V_n^{\otimes k}$, the space of tensors of rank k over a vector space of dimension n, first select an ONB$\{e_1, \ldots, e_n\}$ of V_n. (This also defines a chain $\mathrm{GL}(n) \supset \mathrm{GL}(n-1) \supset \ldots \supset \mathrm{GL}(1)$.) For each k, X will be the set N_n^k of k-tuples

$$(i_1, i_2, \ldots, i_k)$$

where the numbers i_j are drawn from the set N_n of integers from 1 to n. For each $x \in X$, $x = (i_1, i_2, \ldots, i_k)$, define $e_x \in V_n^{\otimes k}$ by

$$e_x = e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_k}.$$

The space $L^2(X)$ with inner product adjusted to

$$\langle \alpha, \beta \rangle = \sum_{x \in X} \overline{\alpha(x)} \beta(x)$$

may be identified with $V_n^{\otimes k}$ by the mapping

$$\beta \leftrightarrow \sum_{x \in X} \beta(x) e_x.$$

Under this identification the inner products coincide, as do the representations of S_k, defined for $V_n^{\otimes k}$ in the usual way, and for $L^2(X)$ via the action of S_k on X given by

$$\sigma(i_1, \ldots, i_k) = (i_{\sigma^{-1}(1)}, \ldots, i_{\sigma^{-1}(k)}) \qquad \sigma \in S_k.$$

(We imagine that S_k is realised as the set of permutations of the integers $\{1, \ldots, k\}$.) We shall denote the representation of S_k on $V_n^{\otimes k}$ by π.

The orbits in X under S_k may be indexed by the set Λ_n^k of partitions of k into n parts. For each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, let i_λ be the element of N_n^k with λ_1 1's, λ_2 2's, $\ldots$, λ_n n's satisfying $i_1 \leq i_2 \leq \ldots \leq i_k$. It is easily seen that the set

$$\{i_\lambda ; \lambda \in \Lambda_n^k\}$$

is a transversal of N_n^k. The stabiliser of i_λ is the subgroup $S_\lambda = S_{\lambda_1} \times \ldots \times S_{\lambda_n}$ of S_k. Under the identification of $L^2(X)$ with $V_n^{\otimes k}$ the summand $L^2(O_\lambda)$ in the decomposition of $L^2(X)$ into invariant subspaces becomes the space of all tensors of weight λ with respect to $\mathrm{GL}(n)$. We can immediately conclude that the tensor representation of S_k restricted to this space is equivalent to the representation of S_k induced by the identity representation ϵ_λ of the subgroup S_λ:

$$\pi \sim \bigoplus_{\lambda \in \Lambda_n^k} \rho^{\epsilon_\lambda}.$$

3.2. *Explicit bases for ρ^{ϵ_λ} and $V_n^{\otimes k}$*

The permutation representations ρ^{ϵ_λ} have been widely studied (Robinson 1961). Henceforth we drop the subscript λ from ϵ_λ: λ may be understood throughout. The multiplicity of the IR μ of S_k in ρ^ϵ is given (as mentioned in the introduction) by the number of Gel'fand patterns of weight λ for the $\mathrm{GL}(n)$ IR with highest weight μ. We shall use the reciprocal bases introduced in § 2 to make this explicit by setting up a 'natural' orthonormal basis for L, the carrier space of ρ^ϵ. The subduced representation reciprocal to μ in ρ^ϵ is ϵ in $\mu|_{S_\lambda}$. Since ϵ is one-dimensional, a basis for the occurrences of ϵ in μ is defined (up to a phase) once the multiplicity of ϵ is resolved. To perform this resolution, we use a well-studied technique for forming a non-standard basis for IR's of S_k.

Fix λ and μ, let $\nu_i = \sum_{j=1}^i \lambda_j$ and consider the chain of subgroups

$$S_k = S_{\nu_n} \supset S_{\nu_{n-1}} \supset \ldots \supset S_{\nu_1} = S_{\lambda_1}$$

where S_ν is the subgroup corresponding to permutations of the integers 1 to ν in the realisation of S_k as the set of permutations of $\{1, \ldots, k\}$. Set $m_n = (\mu_1, \ldots, \mu_n)$ and for each $i < n$, choose $m_i = (m_{i1}, m_{i2}, \ldots, m_{ii})$ to be a dominant partition of ν_i into i parts. Corresponding to each m_i let W_i be the subspace of V (the carrier space of μ) containing all vectors transforming under S_{ν_i} as the IR m_i; let W_ϵ be the subspace of all vectors in V invariant under S_λ. For any given set $\{m_i;\ i=1,\ldots,n\}$ the intersection

$$W_m=\left[\bigcap_{i=1}^{n} W_i\right]\cap W_\epsilon$$

will be either one-dimensional or zero-dimensional; it will be one-dimensional if and only if the numbers (m_{ij}) satisfy the 'betweenness conditions'

$$m_{ij}\geqslant m_{i-1j}\geqslant m_{ij+1}$$

of a Gel'fand pattern. This is a consequence of the rule (due to Littlewood (1940)) for finding the multiplicity of the IR

$$[m_{i1},\ldots,m_{ii}]\times[\lambda_{i+1},0,\ldots,0]$$

of $S_{\nu_i}\times S_{\lambda_{i+1}}$ contained in the IR

$$[m_{i+1,1},\ldots,m_{i+1,i+1}]$$

of $S_{\nu_{i+1}}$; starting with

$$[\lambda_1,\dot{0}]\times[\lambda_2,\dot{0}]$$

one proceeds inductively, increasing i by steps of one, to produce the betweenness conditions. Writing the induction down in tableau form reveals the result clearly. The spaces W_m, as $m=(m_{ij};\ j=1,\ldots,i;\ i=1,\ldots,n)$ runs over the allowed Gel'fand patterns with weight λ and $m_n=\mu$, are one-dimensional, orthogonal if $m\neq m'$, and span the space W_ϵ of vectors in V invariant under S_λ. Choosing a normalised vector from each space we form the set $\{u_m\}$ which is an ONB of W_ϵ.

A relationship between this non-standard basis and the standard Young/Yamanouchi basis for the whole of the IR μ will be useful for establishing the reason why the projectors formed using the latter basis also project onto Gel'fand vectors. A Young/Yamanouchi vector may be specified by a pattern $y=(y_{ij};\ j=1,\ldots,i;\ i=1,\ldots,k)$ corresponding to a $GL(k)$ Gel'fand pattern of weight $[1^k]$, since a standard tableau of weight $[1^k]$ is identical to an ordinary standard Young tableau (Moshinsky 1966, Louck and Biedenharn 1973). If v_y is a normalised vector corresponding to the Young tableau y, it is not hard to see that v_y and u_m will not be orthogonal if and only if $y_{\nu_i}=m_i$ for all i. (The notation y_{ν_i} indicates the set $(y_{\nu_i j};\ j=1,\ldots,\nu_i)$.)

The results of § 2 now show that an ONB of L reciprocal to the set $\{u_m\}$ is given by the set of functions $\{\chi_{my}\}$ with

$$\chi_{my}(g)=(D_\mu|S_\lambda||S_k|^{-1})^{1/2}\langle\mu(g)u_m,v_y\rangle.$$

For definiteness the basis $\{v_r\}$ referred to in § 2 has been taken to be the Young/Yamanouchi basis indexed by the pattern y; the extra factor $|S_\lambda||S_k|^{-1}=\lambda_1!\ldots\lambda_n!/k!$ under the square root sign in the normalisation is there to account for the unnormalised inner product on $L^2(X)$.

The vectors in $V_n^{\otimes k}$ corresponding to the elements χ_{my} in L are conveniently written down using projection operators. Note first that L is the space of complex-valued functions on S_k which are constant on cosets of S_λ. It is not hard to see, using the bijection from S_k/S_λ to $O_\lambda\subset X$, that the function $\psi\in L$ corresponds to the tensor

$$|S_\lambda|^{-1}\sum_{\sigma\in S_k}\psi(\sigma)\pi(\sigma)e_\lambda=|S_\lambda|^{-1}\pi(\psi)e_\lambda$$

where

$$\pi : L^2(\mathrm{S}_k) \to \mathrm{End}(V_n^{\otimes k})$$
$$\psi \mapsto \sum_\sigma \psi(\sigma)\pi(\sigma)$$

is the extension of the representation π from S_k to its group algebra $L^2(\mathrm{S}_k)$ and e_λ is the image of i_λ under the correspondence, i.e. the tensor

$$e_1^{\otimes\lambda 1} \otimes e_2^{\otimes\lambda 2} \otimes \ldots \otimes e_n^{\otimes\lambda n}.$$

When ψ is a matrix element function

$$\psi(\sigma) = D_\mu |\mathrm{S}_k|^{-1} \langle \mu(\sigma) v_r, v_q \rangle$$

between two normalised vectors v_r and v_q, $\pi(\psi)$ is well known (see, for example, Kaplan 1975) to be an operator which projects along the vector v_r and then 'rotates' to the vector v_q. When ψ is the normalised matrix element function above we shall use the notation $P_q{}^\mu{}_r$ for the projection operator $\pi(\psi)$. The desired basis of $V_n^{\otimes k}$ may be then written

$$|{}^{\mu}_{y\times m}\rangle = (|\mathrm{S}_k||\mathrm{S}_\lambda|^{-1} D_\mu^{-1})^{1/2} P_y{}^\mu{}_m e_\lambda.$$

The notation $y \times m$ is used to denote a basis vector of $\mathrm{S}_k \times \mathrm{GL}(n)$. The complete basis is obtained by letting μ run over all dominant partitions of k into not more than $\min(k, n)$ non-zero parts, y run over the Young/Yamanouchi basis of each IR μ of S_k, λ run over the weights occurring in the IR μ of $\mathrm{GL}(n)$ and m run over the Gel'fand patterns of weight λ in the IR μ.

To make contact with the vectors $P_y{}^\mu{}_{y'} e_\lambda$ considered by Patterson and Harter (1976), and also Kaplan (1975) (y' being a second Young/Yamanouchi vector), we return to the observation made earlier that $\langle v_y, u_m \rangle$ is non-zero if and only if $y_{\nu_i} = m_i$ for all i between 1 and n. Let Q be the projector in V for the S_λ IR ϵ; using the Dirac notation $|y\rangle$, etc, for v_y, etc, we have

$$Q = |\mathrm{S}_\lambda|^{-1} \sum_{\sigma\in \mathrm{S}_\lambda} \mu(\sigma) = \sum_m |m\rangle\langle m|.$$

Write m_y for the unique Gel'fand pattern associated with the Young tableau y via λ (i.e. that satisfying $m_{y_i} = y_{\nu_i}$ for all i between 1 and n). Then the uniqueness of m_y shows that

$$\begin{aligned} |\langle m_y | y \rangle|^2 &= \sum_m |\langle m | y \rangle|^2 \\ &= \langle y | Q | y \rangle \\ &= |\mathrm{S}_\lambda|^{-1} \sum_{\sigma \in \mathrm{S}_\lambda} \langle y | \mu(\sigma) y \rangle. \end{aligned}$$

The quantity $\langle m_y | y \rangle$, which may be calculated using this formula and the known matrix elements of the $\sigma \in \mathrm{S}_\lambda$ in a Young/Yamanouchi basis, gives the relation between the vectors $P^\mu_{y' m_y} e_\lambda$ and $P^\mu_{y' y} e_\lambda$:

$$\begin{aligned} P^\mu_{y' y} e_\lambda &= \sum_m \langle y | m \rangle P^\mu_{y' m} e_\lambda \\ &= \langle y | m_y \rangle P^\mu_{y' m_y} e_\lambda. \end{aligned}$$

Therefore the normalised basis may be written

$$|{}^{\mu}_{y\times m_{y'}}\rangle = (|\mathrm{S}_k| D_\mu^{-1} \langle y' | m_{y'} \rangle^{-1})^{1/2} P^\mu_{yy'} e_\lambda.$$

This is the same as the expression found by Patterson and Harter (1976). The discussion above also provides an interpretation of the proportionality between $P^{\mu}_{yy'}e_{\lambda}$ and $P^{\mu}_{yy''}e_{\lambda}$ if $m_{y'} = m_{y''}$, first noted by Goddard (1967).

The expression involving P^{μ}_{ym} may have some advantages over that involving $P^{\mu}_{yy'}$ for performing calculations. The function $\langle\mu(\tau)m|y\rangle$ is constant on cosets τS_{λ} of S_{λ} in S_k; it is actually therefore a function on the space S_k/S_{λ} so that the number of distinct terms occurring in the expansion of P^{μ}_{ym} is smaller than for $P^{\mu}_{yy'}$ by a factor $\lambda_1!\lambda_2!\ldots\lambda_n!$. To calculate $\langle\mu(\tau)m|y\rangle$ for a particular coset representative τ, one may use

$$\langle m|\mu(\tau^{-1})y\rangle = \sum_{y'\sim m} \langle m|y'\rangle\langle y'|\mu(\tau^{-1})y\rangle$$

where the summation is over those Young/Yamanouchi vectors y' such that $m_{y'} = m$. The overlap coefficients $\langle m|y'\rangle$ are evaluated as already described. An advantage of this approach is that one can select the coset representatives τ to simplify the summations as much as possible (although the full summations over the subgroup S_{λ} cannot be avoided).

3.3. *Basis properties under* GL(n)

To confirm that the basis $\{|^{\mu}_{y\times m}\rangle\}$ is in fact the Gel'fand basis arising from the set $e_1, \ldots, e_n$, let P^{μ_i} be the projector onto the IR μ_i of S_{ν_i}. In appendix 2 we prove that

$$P^{\mu}_{ym}P^{\mu_i} = \delta_{\mu_i m_i}P^{\mu}_{ym}.$$

In other words, the range of P^{μ_i} contains the range of P^{μ}_{mm} if μ_i and m_i are equal, and if they are unequal, the ranges are disjoint apart from zero.

Let V_i and $V_i^{\perp}$ be the subspaces of V_n spanned by the vectors $\{e_1, \ldots, e_i\}$ and $\{e_{i+1}, \ldots, e_n\}$, respectively, and for each $v \in V_i^{\perp\otimes(k-\nu_i)}$ form the subspace W_v of $V_n^{\otimes k}$ given by

$$W_v = V_i^{\otimes\nu_i} \otimes v.$$

Each W_v is invariant under $S_{\nu_i}\times GL(i)$ and the representation defined on it by restricting π to this subgroup is equivalent to the natural representation defined on $V_i^{\otimes\nu_i}$. Clearly then $P^{m_i}W_v$ carries the representation $m_i \times m_i$ of $S_{\nu_i}\times GL(i)$. In particular, this is true for

$$v = e_{i+1}^{\otimes\lambda_{i+1}} \otimes \ldots \otimes e_n^{\otimes\lambda_n}$$

and so $P^{m_i}e_{\lambda}$ lies in the IR m_i of GL(i). Since P^{μ}_{ym} commutes with $\pi(GL(n))$, the vector

$$P^{\mu}_{ym}e_{\lambda} = P^{\mu}_{ym}P^{m_i}e_{\lambda}$$

must also lie in the IR m_i of GL(i). Thus the basis $\{|^{\mu}_{y\times m}\rangle\}$ has all the properties of a Gel'fand basis under the subgroups GL(i) of GL(n) for $1 \leqslant i \leqslant n$.

3.4. *An example*

Finally we illustrate the results obtained by means of the simplest non-trivial example. Let μ be the dominant partition (2, 1, 0) of 3 and let λ be the weight (1, 1, 1). There are two Gel'fand patterns corresponding to this pair (μ, λ), namely

$$\begin{matrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{matrix} \quad \text{and} \quad \begin{matrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{matrix}$$

corresponding to the tableaux

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \text{and} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Because λ consists of repeated ones in this case, Young/Yamanouchi and Gel'fand tableaux coincide and the non-standard basis vectors reduce to standard ones. We index each basis simply by the numbers 1 and 2 rather than the full patterns above. Also we write e_{ijk} for $e_i \otimes e_j \otimes e_k$ and $|y \times m\rangle$ for $|^{\mu}_{y\times m}\rangle$. Then

$$|1\times 1\rangle = \frac{1}{\sqrt{12}}(2e_{123} + 2e_{213} - e_{132} - e_{321} - e_{312} - e_{231})$$

$$|1\times 2\rangle = \tfrac{1}{2}(e_{132} - e_{321} + e_{312} - e_{231})$$

$$|2\times 1\rangle = \tfrac{1}{2}(e_{132} - e_{321} - e_{312} + e_{231})$$

$$|2\times 2\rangle = \frac{1}{\sqrt{12}}(2e_{123} - 2e_{213} + e_{132} + e_{321} - e_{312} - e_{231}).$$

It is not hard to check that these vectors are completely reduced (in exactly the way indicated by the appropriate subtableaux) under the chains

$$S_3 \supset S_2 \supset S_1$$

$$GL(3) \supset GL(2) \supset GL(1).$$

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Appendix 1

Antiunitarity of $\hat{T}$.

$$\begin{aligned} \langle T_A, T_B\rangle &= D_\mu^{-1}|G|^{-1}\sum_r \sum_{g\in G} \langle A^*\mu(g^{-1})v_r, B^*\mu(g^{-1})v_r\rangle \\ &= D_\mu^{-1}\,\mathrm{Tr}_\mu(AB^*) \\ &= D_\mu^{-1}\,\mathrm{Tr}_\lambda(B^*A) \\ &= D_\lambda D_\mu^{-1}\langle B, A\rangle. \end{aligned}$$

Appendix 2.

$$P^{\mu}_{ym}P^{\mu_i} = \delta_{\mu_i m_i} P^{\mu}_{ym}.$$

Quite generally, let G be a finite group, H a subgroup, μ, λ IR's of G and H, respectively, and let $\{v_r\}$, $\{w_s\}$ and $\{u_t\}$ be bases as in § 2 where $t = \alpha s$. Let L_G be the group algebra (over $\mathbb{C}$) of G and define a (quasi-right-regular) representation ξ of H on L_G by

$$\xi(h)\psi(g) = \psi(gh) \qquad \psi \in L_G.$$

Let $\chi_{rt}(g) = \langle \mu(g)u_t, v_r \rangle$. Under ξ the set $\{\chi_{rt}\}$ as s runs over the basis of W with μ, r, α and λ fixed ($t = \alpha s$) forms a basis for the IR $\tilde{\lambda}$ of H contragradient to λ. This may be verified in the usual way if one bears in mind that the operators $\{\mu(h); h \in \mathrm{H}\}$ leave invariant for each α the set $\{u_{\alpha s}; s \in W\}$. Therefore if $P^{\tilde{\lambda}'}_{\xi}$ is the projector onto $\tilde{\lambda}'$ in the representation ξ,

$$P^{\tilde{\lambda}'}_{\xi}\chi_{rt} = \delta_{\lambda\lambda'}\chi_{rt}. \tag{A2.1}$$

(Written out in full

$$P^{\tilde{\lambda}'}_{\xi}\chi_{rt}(g) = D_{\tilde{\lambda}'}|\mathrm{H}|^{-1} \sum_{h\in \mathrm{H}} \sum_{s'} \chi_{rt}(gh)\langle \tilde{\lambda}'(h)\tilde{w}_{s'}, \tilde{w}_{s'} \rangle$$

$$= D_{\lambda'}|\mathrm{H}|^{-1} \sum_{h\in \mathrm{H}} \sum_{s'} \chi_{rt}(gh)\langle \lambda'(h^{-1})w_{s'}, w_{s'} \rangle$$

where $\tilde{w}_{s'}$ indicates the vector dual to $w_{s'}$ with respect to $\langle \, , \rangle$. Equation (A2.1) is now a consequence of the orthogonality relations for IR's of H.)

Let π now be an arbitrary representation of G and let $P^{\lambda'}_{\pi}$ be the projector onto the IR λ' in V_π. By expanding everything we easily verify that

$$\pi(\psi)P^{\lambda'}_{\pi} = \pi(P^{\tilde{\lambda}'}_{\xi}\psi) \qquad \psi \in L_G.$$

In particular

$$\pi(\chi_{rt})P^{\lambda'}_{\pi} = \delta_{\lambda\lambda'}\pi(\chi_{rt}).$$

Application of this to the case $\mathrm{G} = \mathrm{S}_k$, $\mathrm{H} = \mathrm{S}_{\nu_i}$, $\mu = \mu$, $\lambda = \mu_i$ yields the desired result.

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